Chapter 1

Eulerian theory of deformable Newtonian lattices

To describe the spatio-temporal evolution of the deformation of a solid lattice, it is first necessary to define a reference system for making space and time measurements in the laboratory. For this, we have the choice between several possible coordinate systems, and we will choose here *the Euler coordinate system* for several reasons which will be explained in detail. Following this choice, it becomes possible to describe how the deformation of a lattice can be characterized by *distortions* and *contortions*. But to quantify these distortions and contortions, it will be necessary to use mathematical objects called *scalars, vectors* and *tensors*. We will therefore try to explain simply these mathematical objects, and why we systematically use *a vector representation of tensors*, which has undeniable advantages over the purely tensorial representation, if only by the possibility of using powerful formalism of a mathematical tool called *vector analysis*. This crucial choice makes it possible to fairly easily obtain equations which ensure the solidity of the lattice, known as *geometro-compatibility equations*, and equations which make it possible to describe the kinetics of the deformation, known as *geometro-kinetic equations*.

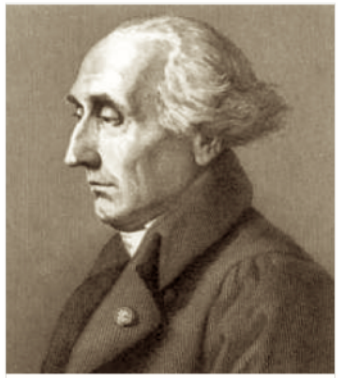
The basic concepts of physics, namely *Newtonian dynamics* and *Eulerian thermokinetics*, can then be introduced in this topological context. With all these ingredients, it then becomes possible to describe a number of specific behaviors of deformable solid lattices, such as *elasticity, anelasticity*, and *self-diffusion*.

Coordinate systems to describe the deformation of a medium

If an observer, who we will call hereafter *Great Observer* ***GO***, wishes to describe in his laboratory the evolution of a certain continuous medium which moves in space by translation and rotation, and which, moreover, can deform at over time (figure 1.1), it must first define the kinetic behavior of the medium. Taking as *a basic axiom* that the evolution of the medium in space and time satisfies *the principle of additivity of velocities*, namely that the velocity of an object moving at velocity v_1 with respect to another moving object at velocity v_2 in the laboratory will have a velocity $v_1 + v_2$ in the laboratory, we will then have to deal with kinetics satisfying *the transformation of Galileo*, and called *Galilean kinetics*. In this case, the observer ***GO*** can describe this spatio-temporal evolution on the basis of an absolute reference system placed in his laboratory. This frame of reference is composed of *an orthonormal Euclidean coordinate system* $\boldsymbol{O}x_1x_2x_3$, that is to say of three rules of unit length, oriented perpendicularly to each other and represented by three arrows which are called the base vectors $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ of the coordinate system, and of a universal clock ensuring that time t is measured identically everywhere in the laboratory (figures 1.1 and 1.2).

To describe the spatio-temporal evolution of a deformable continuous medium, the observer then has several possibilities among which *the Lagrange coordinate system*, used to describe

the deformation of solids, and *the Euler coordinate system*, used in general to describe the hydrodynamics of fluids.


Joseph Louis Lagrange
(1736-1813)

To simply and completely describe the spatio-temporal evolution of a solid continuous medium, the observer ***GO*** can use *the Lagrange coordinate system*. First, it performs a marking of the solid material medium at the initial instant $t = 0$ using a grid of points $\boldsymbol{P}_0$.

Then, he can define a stationary reference frame $\boldsymbol{O}x_1x_2x_3$ in his laboratory. By providing this fixed frame $\boldsymbol{O}x_1x_2x_3$ with unit length rules $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$, and by orienting it judiciously with respect to the initial position of the medium at the instant $t = 0$, it can measure the position of all the points $\boldsymbol{P}_0$ of the medium at the initial instant $t = 0$ using of arrows, vectors[1] $\vec{r}$.

At a point $t > 0$ in time, a point $\boldsymbol{P}_0$ in the middle will have moved to $\boldsymbol{P}$, and the observer can then connect the point $\boldsymbol{P}_0$ to the point $\boldsymbol{P}$ using an arrow, the vector $\vec{u}$ which is called *the displacement vector* of point $\boldsymbol{P}_0$. As this vector depends on the initial position $\vec{r}$ of the point $\boldsymbol{P}_0$ and time t, the set of vectors $\vec{u}(\vec{r},t)$ identifying all the points of the medium is called *the displacement field* of the medium in Lagrange coordinates.

The Lagrange coordinate system is therefore based on the description of the evolution in space and time of *the vectors* $\vec{u}(\vec{r},t)$ *of the displacement field* defined above, knowing the

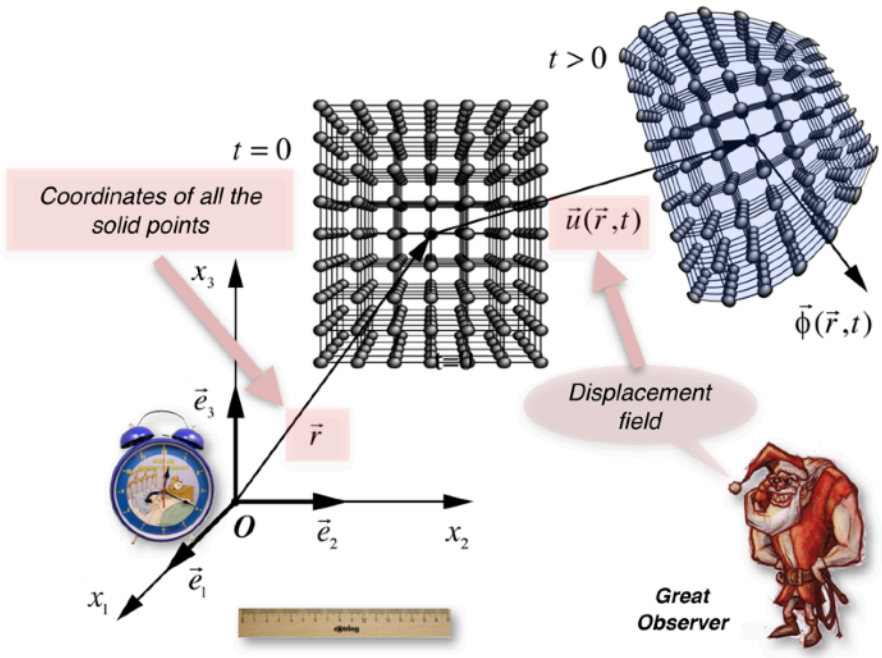
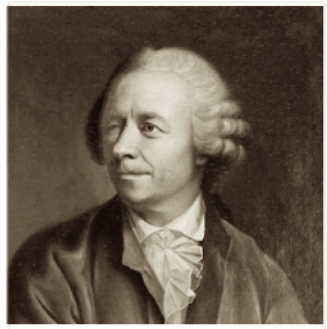


Figure 1.1 - *The Lagrange coordinates*

[1] ***Vector:*** a vector is a mathematical object corresponding to an arrow oriented in space. A vector actually represents a physical quantity described by three numbers which correspond respectively to the lengths of the three projections of the arrow on the axes $\boldsymbol{O}x_1x_2x_3$ of the coordinate system. We speak of a vector field when a vector physical quantity takes different values at all points in space and over time.



Leonhard Euler
(1707-1783)

coordinates $\vec{r}$ of all the points of the initial solid in the fixed frame $\boldsymbol{O}x_1x_2x_3$ of the laboratory of the observer, as illustrated in figure 1.1.

By using previously the expression of *continuous medium*, one appealed to an intuitive concept, namely that a medium presents, on the *macroscopic scale* where it is observed, neither discontinuous structure in the static state, nor appearance of discontinuities, such as tears, local ruptures or cavity formations during its spatio-temporal evolution.

From the macroscopic observation of the behavior of the medium, and in particular of *the continuity of the displacement field* $\vec{u}$, it is possible to attribute qualifiers to the medium observed. If the medium presents a perfectly continuous displacement field during its spatio-temporal evolution, it is qualified as *solid medium*. It then has the macroscopic property of having its own form which is difficult to modify.

If, on the other hand, the medium has a discontinuous displacement field $\vec{u}$, forming over time an inextricable entanglement, it is qualified as a *fluid medium*. This has the macroscopic property of flowing and must therefore be kept in a container whose shape it follows. In this case, the displacement field $\vec{u}$ of the Lagrange coordinates loses all physical significance, and only *the vector velocity* $\vec{\phi}(\vec{r},t)$ of the fluid located at the instant t at the space coordinate $\vec{r}$ of the absolute reference frame retains a physical meaning. This definition of the movements of the medium by the observer ***GO*** is called *the Euler coordinate system*.

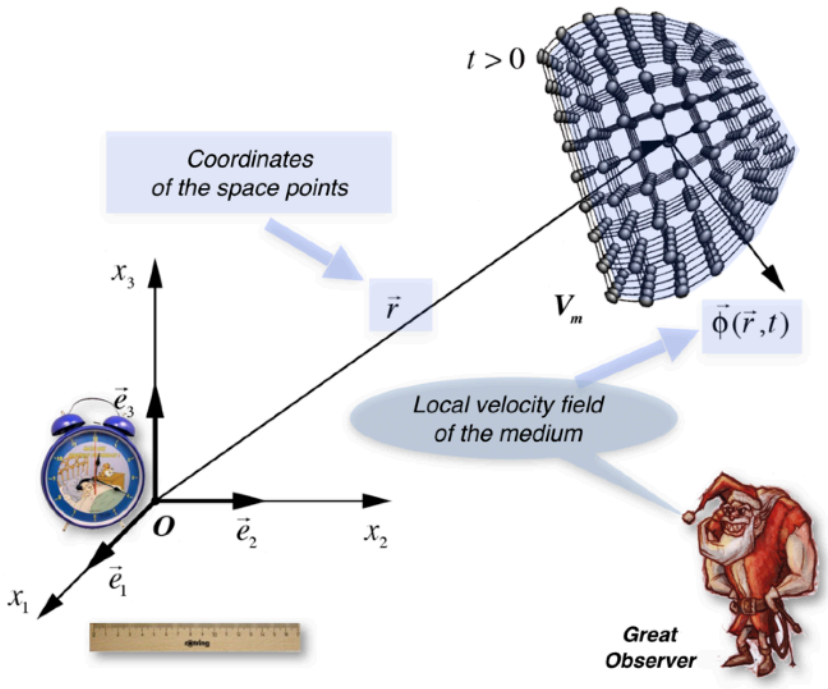


Figure 1.2 - *The Euler coordinates*

The Euler coordinate system is therefore based on the description of the evolution in space and time of the vectors of *the velocity field* $\vec{\phi}(\vec{r},t)$ of the points of the fluid or solid medium located at the space $\vec{r}$ and time t coordinates in the absolute coordinate system $\boldsymbol{O}x_1x_2x_3$ *of* the observer ***GO*** laboratory (figure 1.2).

What is a solid lattice?

The concept of continuous medium defined above only applies when this medium is observed *on a macroscopic scale*. Indeed, an enlargement of the same medium on *a sufficiently microscopic scale* will reveal a discontinuous collection of objects, to which we will attribute in the following *the generic name of particles* (for example corpuscles, atoms, molecules, etc.). We logically conclude that the global phenomenological properties observed at the macroscopic scale where the medium appears continuous are in fact *statistical effects* resulting from the large number of particles interacting with each other at the microscopic scale.

The enlargement of the medium also makes it possible to define certain important microscopic characteristics of the medium, such as *its structure*, that is to say the way in which the particles which compose it are assembled, and *its chemical composition*, that is to say ie the nature of the particles that compose it.

A continuous medium will be said to be solid when, on the microscopic scale, it corresponds to a collection of particles such that the identity of the nearest neighbors of a given particle does not change over time. In other words, each particle is connected to its closest neighbors by *elastic bonds* which prevent it from moving at a great distance from these. Consequently, only relative movements at a short distance from its closest neighbors are allowed to it via the elasticity of the bonds. Under the effect of these bonds, it is said that the particles then form a *solid lattice*.

It is possible to define different classes of solid lattices, according to the arrangement of the particles with respect to each other. If the arrangement of the particles presents a well-established order, which is repeated at great distance by translation of an elementary cell, we speak of a lattice of *crystal structure*. For example, the three-dimensional lattices drawn in figures 1.1 and 1.2 are obtained by the translation of a cubic unit cell, and they present a perfect order both at long distance and at short distance. The same is true of the two-dimensional lattice shown in figure 1.3a, which is obtained by translation in space of a hexagonal cell.

Certain solid lattices can present arrangements of particles having no order at long distance, but only *a certain order at short distance*. We speak in this case of an *amorphous lattice structure*. The two-dimensional example shown in figure 1.3 (b) represents an amorphous lattice of particles, obtained by tiling the surface with pentagons, hexagons and irregular heptagons, whose sides have a fixed length. The short-range order of the amorphous lattice is reflected in the fact that each of the particles has exactly three closest neighbors.

There can also exist solid lattices whose arrangement of particles does not present an order by translation at long distance, but *a certain order by rotation*. In this case, we are talking about *a lattice of quasi-crystalline structure*, the example of which in figure 1.3 (c) clearly shows the absence of order by long-distance translation. This lattice is obtained by two-dimensional tiling using two different types of rhombuses with different angles at the top (in this case, we speak of *Penrose tiling*). At first glance, this lattice seems amorphous. But the more detailed analysis of

figure 1.3 (d) shows that the particles are aligned on parallel lines between them. The distances between particles aligned on a straight line, as well as the distances between parallel straight lines, are not regular. There are in fact five preferred directions for the orientation of these alignments of particles. This two-dimensional quasi-crystalline structure therefore has a form of symmetry *by rotation of order five*, which is prohibited in the case of a crystalline structure obtained by translation of a basic pattern.

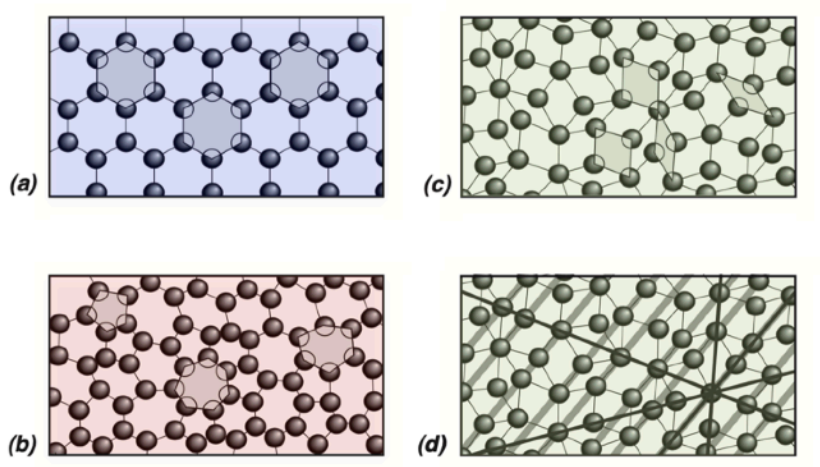


Figure 1.3 - *Two-dimensional crystalline lattice* ***(a)****, amorphous lattice* ***(b)*** *and quasi-crystalline lattice* ***(c)*** *,* ***(d)***

The examples given in figure 1.3 are two-dimensional representations. It is therefore still necessary to generalize these concepts to three-dimensional space. In three dimensions, the crystal lattices are formed by the translation of a three-dimensional elementary cell which is called *the unit cell of the lattice* (figure 1.4). The crystal lattices can be classified according to the operations of rotational symmetry, reflection and inversion with respect to a point which it is possible to apply to the elementary unit cell of the lattice. These symmetry operations lead to the existence of fourteen different lattices, called *Bravais lattices*, which are shown in Figure 1.4. These fourteen Bravais lattices can be further broken down into *seven crystalline systems* (triclinic, monoclinic, orthorhombic, tetragonal, rhombohedral, hexagonal and cubic) by considering the way in which space is paved by the elementary unit cell. For some of these seven crystalline systems, there may be different types of lattices (*P, C, I* or *F*), which correspond to the different patterns of filling of the elementary unit cell with particles.

It is interesting to note that the pattern of filling of a unit cell by the particles can lead to different values of the number of alternative sites specific to each unit cell and which can contain a bound particle. For example, in the case of the cubic structures represented in figure 1.4, the number of sites specific to a unit cell is 1 for the simple cubic system, 2 for the centered cubic system and 4 for the cubic system with centered faces.

In the case of non-ordered solid media, such as amorphous media, quasi-crystalline media or polycrystalline media with very fine grains, the notion of lattice unit cell no longer has any meaning. On the other hand, the concept of lattice site retains a precise physical meaning, even if there is no single elementary unit cell.

In the case of an ordered solid lattice, structural defects may appear in the regular assembly of the lattice particles. These structural defects have various origins, such as irregularities in the chemical species of the objects making up the lattice, or topological singularities, that is to say irregularities in the topological structure of the lattice, such as dislocations or disclinations which will be discussed later, and they can be classified into point, linear or planar defects according to their topology in the lattice.

It is also by observations of the dynamics at the microscopic scale, during the macroscopic spatio-temporal evolution of the lattice, that it will be possible to understand the objective reasons for certain macroscopic behaviors. For example, we will see that there are close links between the macroscopic deformation properties of ordered lattices and the lattice unit cell distortions induced by the presence of mobile topological singularities within the lattice, such as dislocations or disclinations.

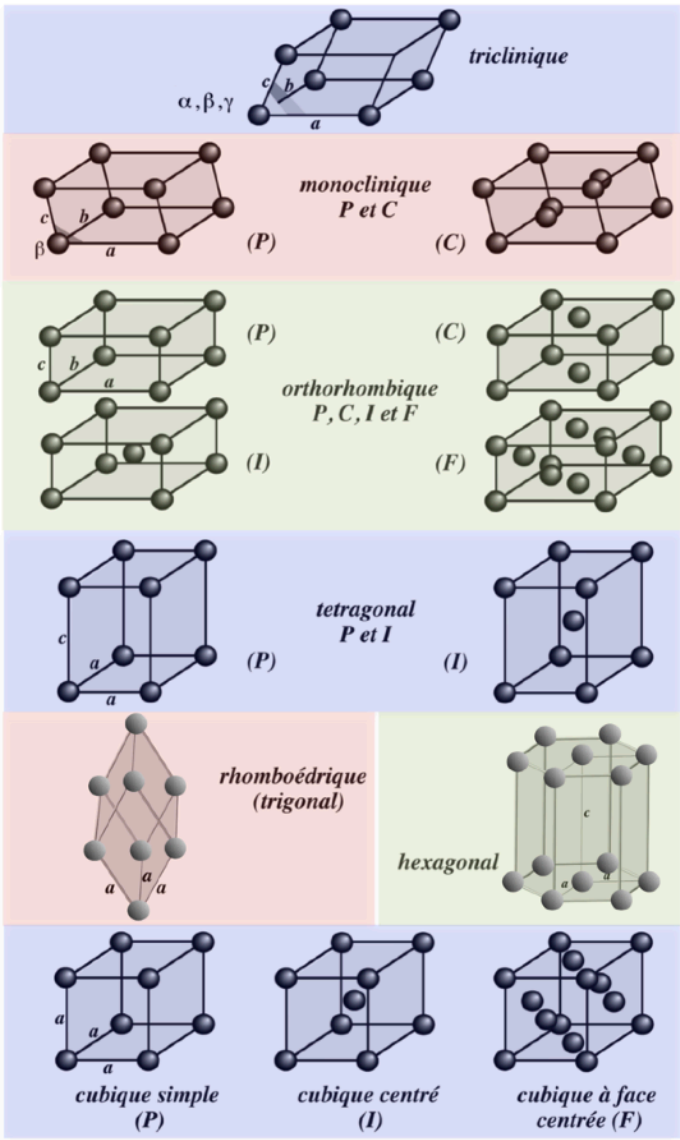


Figure 1.4 - *The seven crystalline systems and the fourteen Bravais lattices*

In conclusion, a complete description of the spatio-temporal evolution of a lattice which can be considered as continuous on the macroscopic scale cannot absolutely do without the description of the phenomena occurring on the microscopic scale. The search for a theory describing the macroscopic spatio-temporal evolution of a deformable continuous lattice must therefore be based on the definition of mean macroscopic fields (scalar, vector and tensorial in nature) deduced from a statistical description of the dynamics at the microscopic scale of a multitude of objects interacting with each other.

To describe a solid lattice, it is therefore perfectly possible to use the Lagrange coordinate system or the Euler coordinate system. However, using Lagrange coordinates to describe deformable solids presents a number of inherent difficulties. From a mathematical point of view, mathematical objects (tensors) describing the deformations of a continuous solid in Lagrange coordinates are always very complicated (of order greater than one in the spatial derivatives of the components of the displacement field), which leads to a mathematical formalism very difficult to manage when a solid presents strong distortions (deformations and rotations). To these mathematical difficulties are added physical difficulties when it comes to introducing certain known properties of solids. Indeed, the Lagrange coordinate system becomes practically unusable, for example when it is necessary to describe the temporal evolution of the microscopic structure of a solid lattice (phase transitions) and of its structural defects (point defects, dislocations, disclinations, joints, etc.), or if it is necessary to introduce certain physical properties of the medium (thermal, electrical, magnetic, chemical, etc.) resulting in the existence in real space of scalar, vector or tensorial fields. Given the complexity of the calculations obtained in the case of the Lagrange coordinate system, which is however generally used to describe solids, it was desirable to try to develop the description of solids using the Euler coordinate system, which is generally used to describe fluids. This approach to deformable solids by Euler coordinates, which is ultimately much simpler and much more rigorous than that obtained with Lagrange coordinates, was developed and published in 2013 in the book *"Eulerian theory of deformable media"*[2] .

Definition of local quantities in Euler coordinates

In the case of a collection of particles in space in the liquid or solid state, each particle i has its own velocity represented by a vector $\vec{v}_i$ (figure 1.5). To determine *an average local velocity* of the particles, you need to fix a small volume element $\boldsymbol{V}_f$ centered on the space coordinate $\vec{r}$, then measure the velocities $\vec{v}_i$ of all the particles contained in the fixed volume $\boldsymbol{V}_f$.

If the instantaneous number of particles in this volume $\boldsymbol{V}_f$ is equal to $\boldsymbol{N}$, and that $\boldsymbol{N}$ is sufficiently large, the average velocity $\vec{\phi}$ at the place $\vec{r}$ and at the time t is defined by the average of the velocities $\vec{v}_i$ taken on all the particles contained in $\boldsymbol{V}_f$.

If an average velocity $\vec{\phi}$ other than zero is measured, it also means that, for each particle, it is possible to find *a fluctuation* $\Delta\vec{v}_i$ *at average velocity* $\vec{\phi}$ by the relation $\Delta\vec{v}_i = \vec{v}_i - \vec{\phi}$.

---

[2] *G. Gremaud, "Théorie eulérienne des milieux déformables – charges de dislocation et désinclinaison dans les solides", Presses polytechniques et universitaires romandes (PPUR), Lausanne (Switzerland) 2013, 751 pages, ISBN 978-2-88074-964-4*
G. Gremaud, "Eulerian theory of newtonian deformable lattices – dislocation and disclination charges in solids", Amazon, Charleston (USA) 2016, 312 pages, ISBN 978-2-8399-1943-2

In the case of a solid lattice, the existence of an average velocity $\vec{\phi}(\vec{r},t)$ other than zero implies that the solid lattice of particles is subjected to *a collective movement*. The velocity $\vec{\phi}(\vec{r},t)$ therefore represents the average local velocity of movement of the particles linked to the lattice sites, therefore the average velocity of the lattice sites (figure 1.2), while the $\Delta\vec{v}_i$ are the velocity fluctuations of the particles linked to the lattice around each of these sites. For example, in a real solid, such fluctuations are due to the disordered movements of *the thermal agitation* of the particles, which are in direct relation with *the temperature of the medium*.

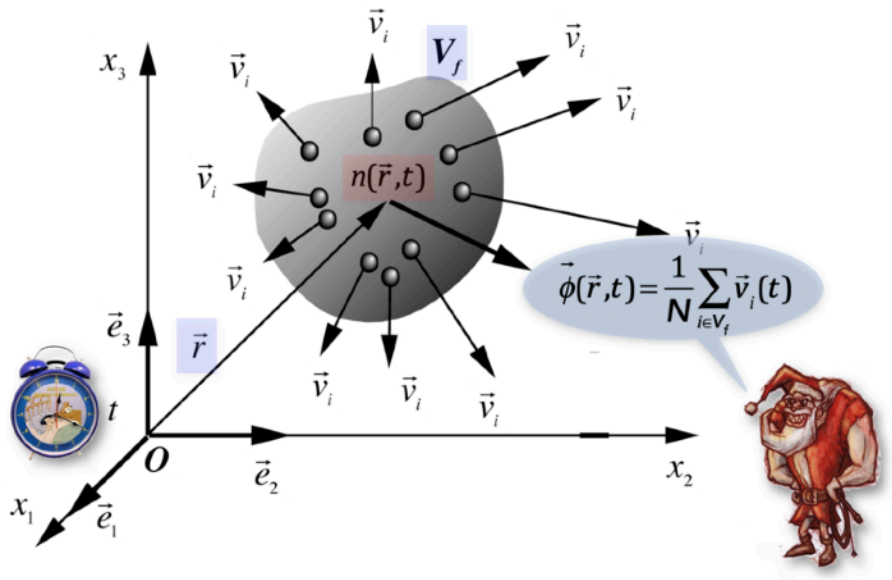


Figure 1.5 - *Velocity and average local densities of a medium*

Apart from the average local velocity $\vec{\phi}(\vec{r},t)$ of a solid lattice, there is another quantity which will be called upon to play a fundamental role in Euler coordinates: it is *the volume density* of elementary substitutional sites of the lattice, which will be written with the symbol n, and which represents *the number of lattice sites contained in the volume unit* of the lattice. In the case of figure 1.5, this number n represents then the number of particles contained in the volume V_f when one chooses a volume V_f equal to the unit. This choice implies then to define all the physical quantities characterizing the solid lattice like average values taken on each site in the lattice. It is also clear that, in the case of unordered lattices, the quantity n can also be related to the density of elementary sites of the unordered lattice.

The *volume density* $n(\vec{r},t)$ of substitutional sites of the solid lattice presents a direct link with the notion of volume expansion of the medium, since $n \to 0$ for intense expansions and $n \to \infty$ for intense contractions. This notion of expansion of the lattice volume can be expressed even better by introducing a quantity v defined as the inverse of n, that is to say $v = 1/n$. Indeed, this quantity v has for dimension a volume. It represents *the average volume occupied by a solid lattice site*. This volume v translates well the notion of volume expansion of the medium, since $v \to \infty$ for intense expansions and $v \to 0$ for intense contractions.

But it is even more interesting to introduce a dimensionless value using *the natural logarithm* of v thanks to the relation $\tau = -\ln(n/n_0) = \ln(v/v_0)$. There appears then *a dimensionless*

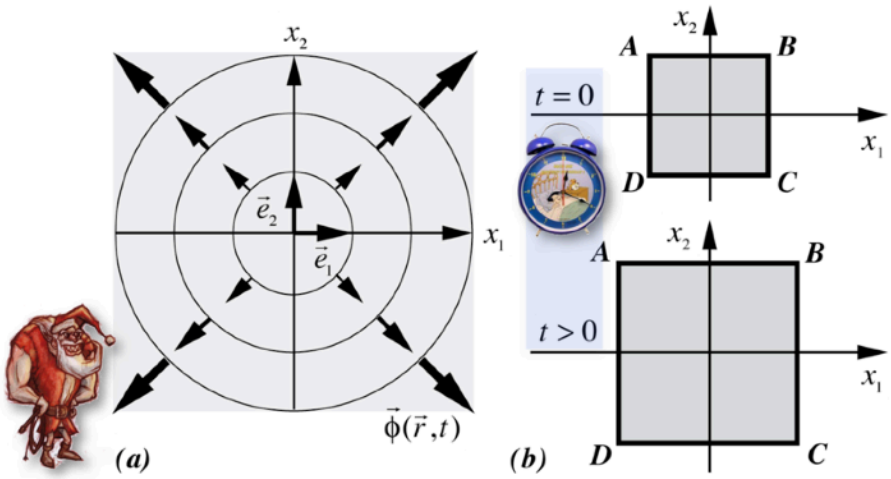


Figure 1.6 - *Volumic expansion of the medium*

scalar[3] $\tau(\vec{r},t)$ which will be called *scalar of volume expansion*, depending on the coordinates of space $\vec{r}$ and time t within the lattice, and this time perfectly measuring the notion of volume expansion of the lattice, since $\tau \to \infty$ for intense expansions (when $v \to \infty$), $\tau \to -\infty$ for intense contractions (when $v \to 0$) and $\tau \to 0$ when $v \to v_0$. By the construction of the scalar τ, the constants v_0 and n_0 introduced here can be adjusted so that the scalar τ is zero when the lattice is in the state of expansion of rest. Figure 1.6 illustrates an example of velocity fields which leads to a uniform volume expansion of the medium which goes from a certain value of τ at the instant $t=0$ to a greater value of τ at the instant $t>0$, without the volume expansion τ depending on the place where it is measured.

Distortions of a solid lattice

In the presence of a non-zero velocity field $\vec{\phi}(\vec{r},t)$ in space, a lattice can present movements which do not lead to any deformation, such as *the global translation* of the medium (figure 1.7a) or *the global rotation* of the medium (figure 1.7b).

But there can also exist non-zero velocity fields $\vec{\phi}(\vec{r},t)$ in space which lead to *real deformations* of the medium, such as for example *the volume expansion* of the medium described above (figure 1.6), *the pure shear* with zero volume expansion of the medium (figure 1.8) or *the zero volume expansion elongation* of the medium (figure 1.9). There may also appear much more complicated velocity fields leading for example to local rotations with non-uniform shearing of the medium (Figure 1.10a) or to non-uniform expansions of the medium which lead to an expansion which depends not only on time, but also space coordinates (figure 1.10b).

In the presence of a non-homogeneous velocity field $\vec{\phi}(\vec{r},t)$ in space, a lattice can therefore present, in addition to a global translation and a global rotation, movements corresponding to all kinds of deformation. To explain in detail the behaviors of global and local rotation and the

---

[3] ***Scalar:*** a scalar is a mathematical object corresponding to a physical quantity described by a single number. We speak of a scalar field when a scalar physical quantity takes different values at all points in space and over time.

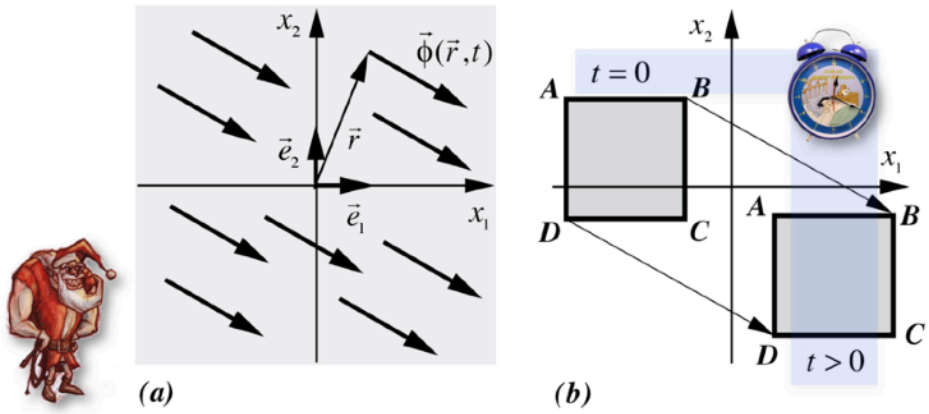


Figure 1.7a - *Global translation of the medium*

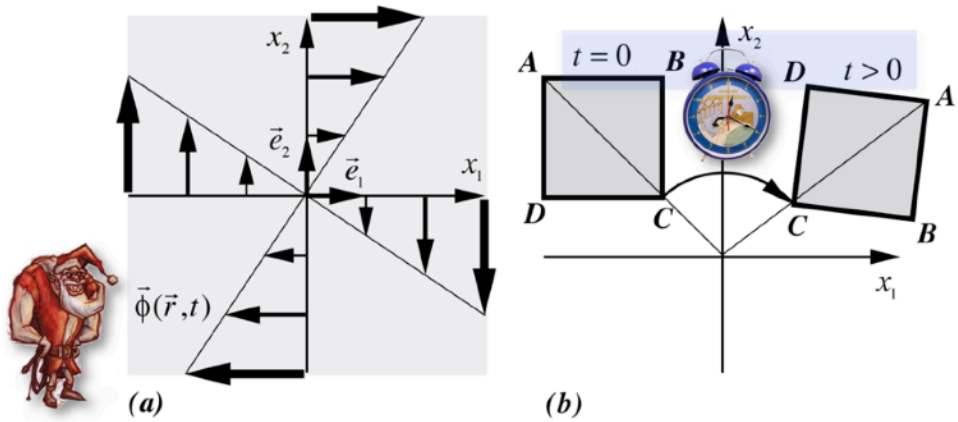


Figure 1.7b - *Global rotation of the medium*

behaviors of deformations, which will be called generically *distortions of the lattice*, it is necessary to introduce topological quantities having to translate these distortions.

An elegant way of proceeding in Euler coordinates is to note that the scalar volume expansion field τ defined above is already a scalar topological quantity which perfectly describes the volume expansion of the medium. We can then show that this scalar is in fact deductible from a more complicated topological quantity, namely *a tensor*[4] *of second order* which is called *the tensor of distortion* β_{ij}. A tensor of order two is a mathematical object represented by an array 3 of 3 of nine different numbers (figure 1.11).

These nine numbers are then sufficient to describe perfectly *the set of global and local rotations and deformations* of the solid lattice. But as the manipulation of this tensor is mathematically quite complicated, and especially since it generally camouflages the real physical behavior of the medium, we choose an entirely original way of representing it in the

[4] ***Second order tensor:*** second order tensor is a mathematical object represented by an array 3 of 3 of nine different numbers. A tensor of order two actually represents a physical quantity described by nine numbers. It can be very convenient to represent a tensor of order two using three vectors in the Euler coordinate system. We speak of a tensor field of order two when a tensorial physical quantity takes different values at all points in space and over time.

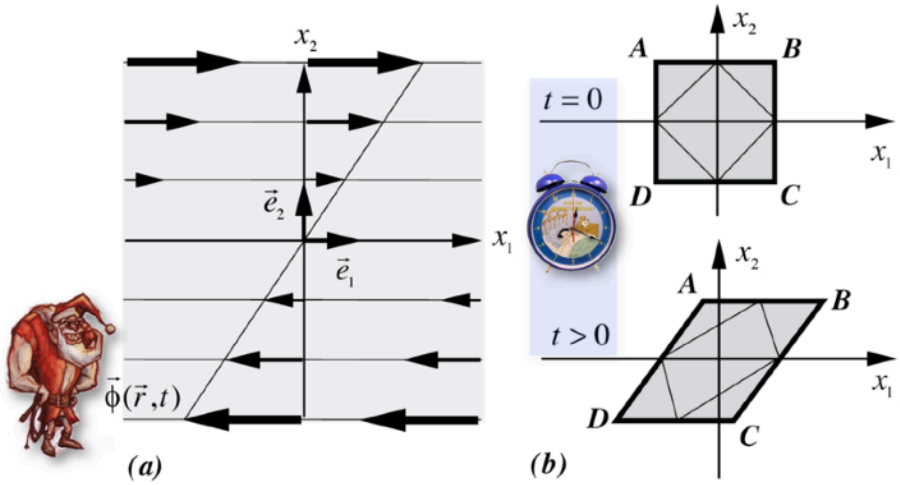


Figure 1.8 - *Pure shear of the medium*

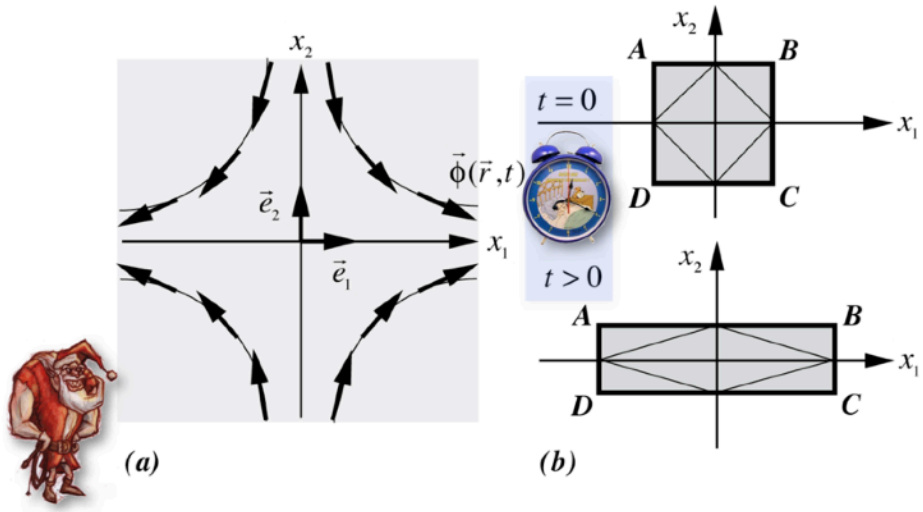


Figure 1.9 - *Elongation of the medium without volumic expansion*

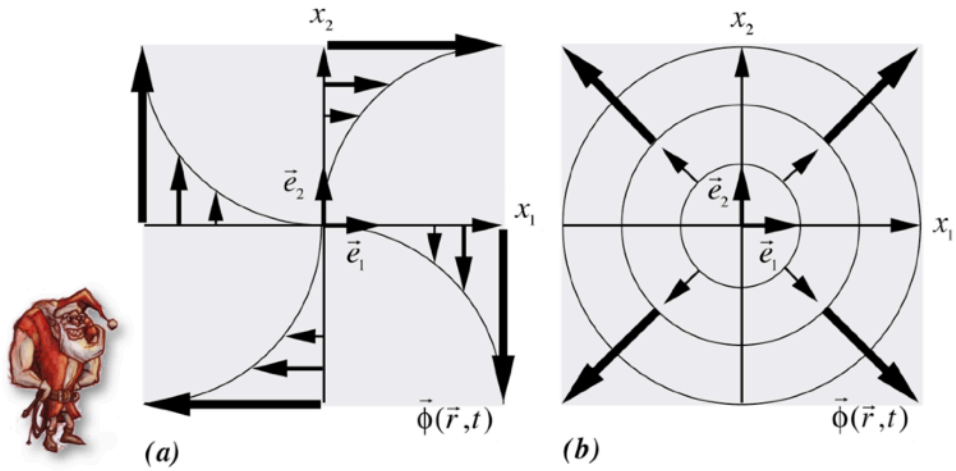


Figure 1.10 - *Non-uniform local rotations* ***(a)*** *and expansions* ***(b)*** *of the medium*

form of three vectors. Thus, the tensor field of lattice distortion β_{ij} will be represented for convenience by *a field of three vectors* $\vec{\beta}_1$, $\vec{\beta}_2$, $\vec{\beta}_3$, remembering that a vector is an arrow oriented in space composed of three numbers. This *vectorial representation of the tensor fields* is completely original and extremely powerful, because it makes it possible mathematically to call upon the spatial operators of *the vector analysis*, and it then considerably simplifies the physical interpretation of the tensor fields.

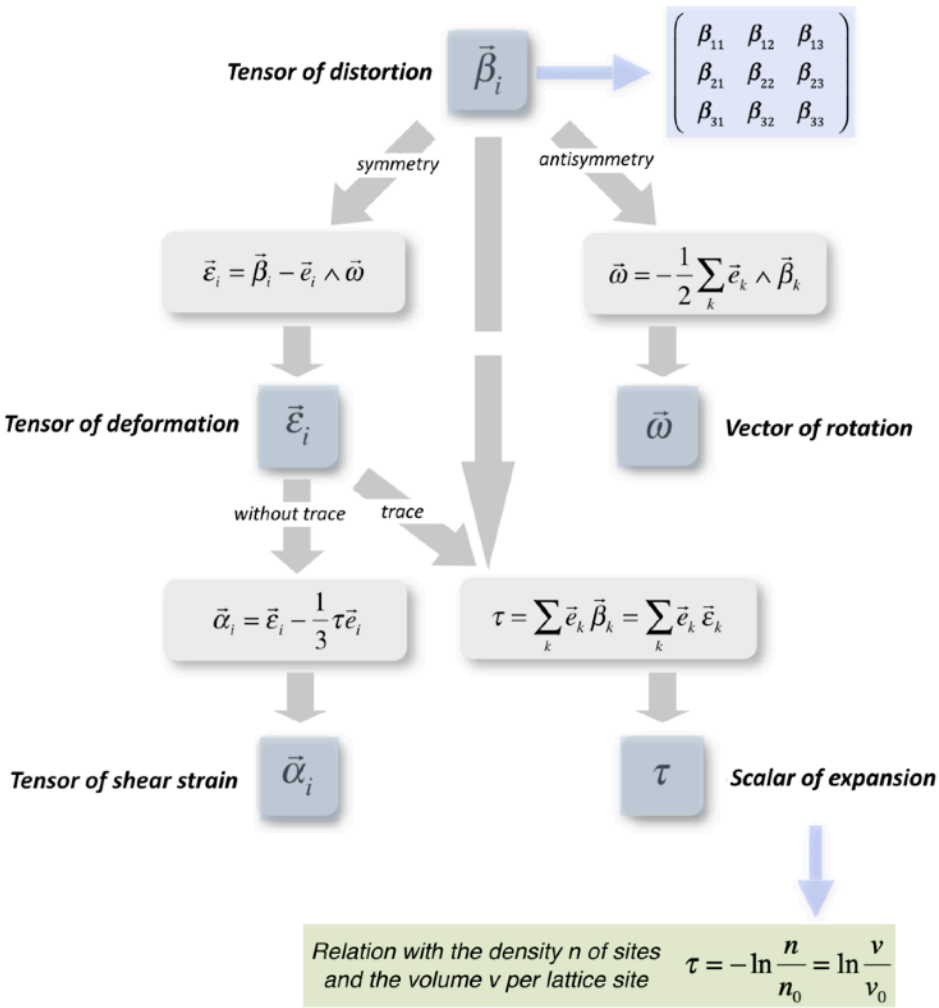


Figure 1.11 - *The decompositions of the tensor of distortion*

It is possible to apply *symmetry operations* on *the distortion tensor* $\vec{\beta}_i$ to extract the details of the rotations and the deformations of the medium. The operation of carrying out the sum of the diagonal elements of the tensor, namely $\beta_{11} + \beta_{22} + \beta_{33}$, provides *a scalar* called *the trace of the tensor* and which is in fact nothing other than the scalar of volume expansion τ .

The operation of taking the symmetrical part of the distortion tensor provides a symmetric tensor of second order $\vec{\varepsilon}_i$, called *the strain tensor*, which represents all the deformations of the medium, but without the global rotations of the medium.

The operation of taking the anti-symmetrical part of the distortion tensor provides an axial vector $\vec{\omega}$, called *the rotation vector*, which represents all the local and global rotations within

the medium.

Finally, by removing its trace from the strain tensor $\vec{\varepsilon}_i$, we obtain a symmetric tensor of second order without trace $\vec{\alpha}_i$, called *the shear tensor*, which represents the set of shears of the medium. These operations of symmetry are described in figure 1.11, where, for the fine connoisseurs of vectorial computation, one made explicitly appear the mathematical operations of symmetry used, as well as the relation existing between the scalar of volume expansion, the density of sites of the network and the volume per network site.

Geometro-kinetics of a lattice in Euler coordinates

In Euler coordinates, we describe the evolution of the solid network in space and time using *the vectors of the velocity field* $\vec{\phi}(\vec{r},t)$ of the points of the lattice located at the space and time coordinates $\vec{r}$ and t in the absolute reference frame $\boldsymbol{O}x_1x_2x_3$ of the laboratory of the observer ***GO*** (Figure 1.2). However, if there is a non-homogeneous velocity field within the lattice, there must necessarily appear a spatio-temporal evolution of the distortions within this lattice. In Euler coordinates, the relationships between the velocity field $\vec{\phi}(\vec{r},t)$ and the evolution of the distortion tensor $\vec{\beta}_i$, the rotation vector $\vec{\omega}$ and the volume expansion scalar τ will be called *the geometro-kinetic equations*. These equations are shown in figure 1.12.

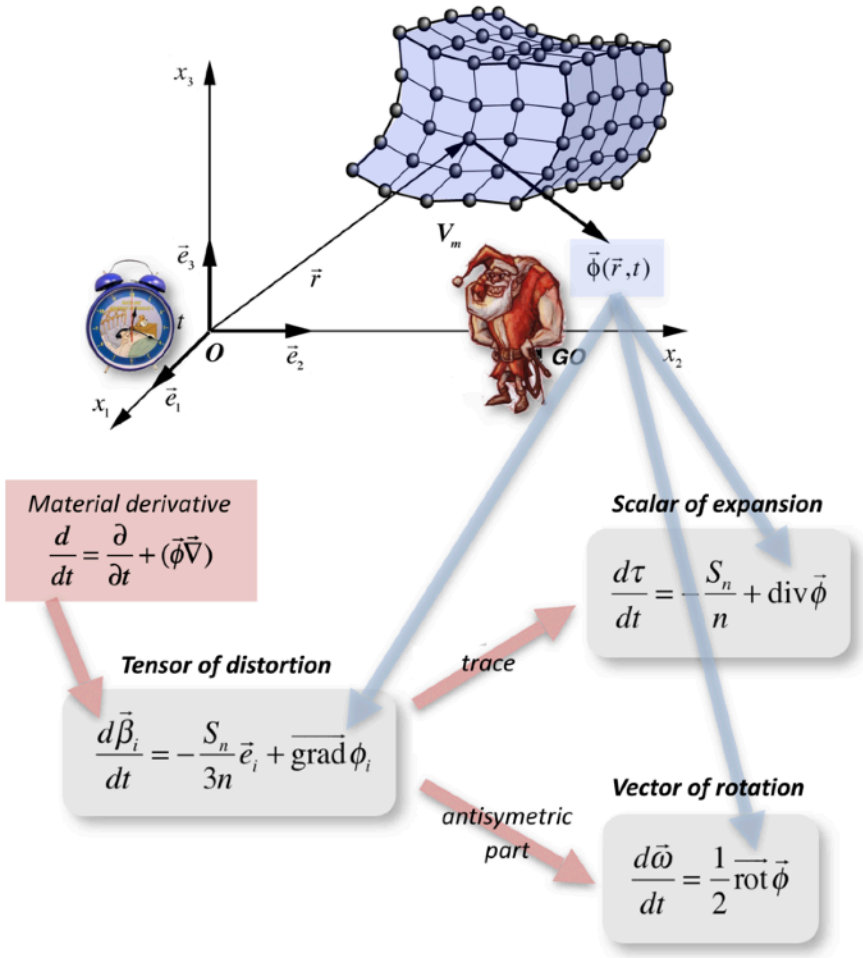


Figure 1.12 - *The geometro-kinetic equations*

Geometro-kinetic equations actually link *the temporal variations of the distortions* of the solid, which are calculated along the trajectory of the particles of the medium using *a mathematical operator of time* called *the material derivative*[5], with *the spatial variations of the velocity field* $\vec{\phi}(\vec{r},t)$ of the medium, which are calculated using mathematical space operators from vector analysis applied to the velocity field, and which are called *the gradient*[6] in the case of the distortion tensor, *the curl*[7] in the case of the rotation vector and *the divergence*[8] in the case of the scalar of volume expansion. In these geometro-kinetic equations, there also appears a quantity S_n which corresponds to the possibility of the existence of a mechanism which can create or destroy lattice sites. This quantity S_n is called *the source of lattice sites*, and corresponds to the number of lattice sites created or destroyed per unit of time and per unit of volume of the lattice. The mechanisms leading to such sources of lattice sites will be discussed later.

Geometro-compatibility of lattice distortions

We have previously shown that the description of solids in Lagrange coordinates is characterized by a displacement field $\vec{u}(\vec{r},t)$. Indeed, in Lagrange coordinates (figure 1.1), the solid is described by a vector $\vec{r}$ locating the initial position of all its points in the reference frame $\boldsymbol{O}x_1x_2x_3$ of the observer ***GO***. The Lagrange displacement vector field then makes it possible to locate in space at the instant t the position of all the points of the solid initially located at the coordinate $\vec{r}$ in the coordinate system $\boldsymbol{O}x_1x_2x_3$.

It is intuitively clear that the description of the distortions of a solid in Euler coordinates should also make it possible to find such a displacement field. Indeed, in Euler coordinates (figure 1.2), the deformed solid is described at instant t in the absolute frame of reference of the observer ***GO***. For a point $\boldsymbol{A}$ of the solid located at the coordinate $\vec{r}$ of this frame of reference it must be possible to define a displacement vector $\vec{u}_E(\vec{r},t)$ which connects this point $\boldsymbol{A}$ to the place $\boldsymbol{A}'$ *where the same point* $\boldsymbol{A}$ *of the solid was located at the initial instant* $t=0$.

There is a close link between the particle time derivative of the distortion tensor $\vec{\beta}_i$ and the gradient of the components of the velocity field $\vec{\phi}(\vec{r},t)$ as shown in the geometro-kinetic equation reported in figure 1.12. However, the velocity field $\vec{\phi}(\vec{r},t)$ itself must be closely related to the temporal variation, called the temporal derivative, of the displacement field $\vec{u}_E(\vec{r},t)$. It is therefore deduced that there must necessarily be a close link between the distortion tensor $\vec{\beta}_i$ and the gradient of the displacement field $\vec{u}_E$ if the latter exists, and that the distortion tensor $\vec{\beta}_i$ is very probably *the gradient tensor* $\vec{\beta}_i = -\overrightarrow{\mathrm{grad}}\, u_{Ei}$ of the components of the displacement field $\vec{u}_E$.

---

[5] ***Material derivative operator:*** it is a *mathematical operator of time* allowing to calculate the temporal variations of a physical quantity *along the trajectory of the particles* of a medium (see glossary).

[6] ***Gradient operator:*** the *gradient of a scalar field* f is a mathematical space operator providing *a vector field* $\vec{u}$ (see glossary).

[7] ***Curl operator:*** the *curl of a vector field* $\vec{u}$ is a mathematical space operator providing *another vector field* $\vec{v}$ (see glossary).

[8] ***Divergence operator:*** the *divergence of a vector field* $\vec{u}$ is a mathematical space operator providing *a scalar field* g (see glossary).

Even if this reasoning seems complicated, it spontaneously leads to a mathematical condition on the distortion tensor $\vec{\beta}_i$ so that there really exists a displacement field $\vec{u}_E$. Indeed, in vector analysis, we demonstrate an unavoidable mathematical property, namely that the curl of a gradient is necessarily zero. Consequently, for there really exists a displacement field $\vec{u}_E$ in coordinates of Euler, it is *necessary* that the curl of the tensor of distortion $\vec{\beta}_i$ is null, therefore that $\overrightarrow{\mathrm{rot}}\,\vec{\beta}_i = 0$.

This equation is called *the geometro-compatibility condition* of the distortion tensor $\vec{\beta}_i$, and it ensures that there is indeed a continuous displacement field $\vec{u}_E$ in Euler coordinates. Note again that if we take the trace of the geometro-compatibility equation for $\vec{\beta}_i$, in other words the sum of the diagonal elements of the tensor $\overrightarrow{\mathrm{rot}}\,\vec{\beta}_i$, we find a new geometro-compatibility condition which then applies to the rotation vector $\vec{\omega}$, namely that the divergence of the vector of rotation $\vec{\omega}$ must be null so that there exists a field $\vec{u}_E$ of continuous displacement in coordinates of Euler. These two geometro-compatibility equations are essential to ensure that a solid does not tear and that there do not appear any cavities during its space-time evolution. The various operations performed on the distortion tensors are summarized in figure 1.13.

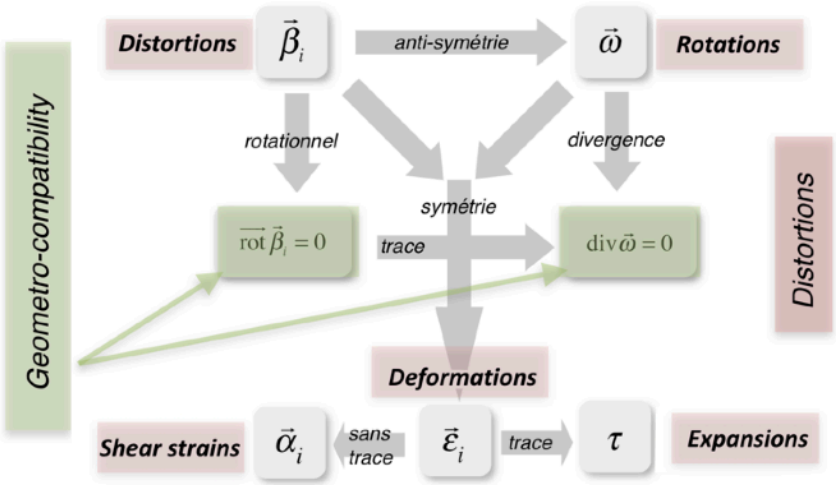


Figure 1.13 - *The complete system of distortions of a solid lattice*

The physical interpretation of these two *geometro-compatibility conditions* can be illustrated quite easily by the drawings shown in figures 1.14 and 1.15. The compatibility condition $\overrightarrow{\mathrm{rot}}\,\vec{\beta}_i = 0$ for the distortion tensor $\vec{\beta}_i$ implies that the displacement field $\vec{u}_E$ has good continuity properties. To show it, it suffices to consider a closed contour $\boldsymbol{C}$ within the medium, and to transfer the vectors of displacement $\vec{u}_E$ along this contour (figure 1.14). If the medium presents a field of distortions satisfying the condition of compatibility $\overrightarrow{\mathrm{rot}}\,\vec{\beta}_i = 0$, the vector of closure $\vec{\boldsymbol{B}}$, called dislocation of the medium, is null, which effectively means from a topological point of view, that *there are no discontinuities of displacements*, called *dislocations*, in the medium.

The existence of a displacement field $\vec{u}_E$ without discontinuities makes it possible to ensure *the topological connectivity* of the medium, that is to say from a physical point of view the fact

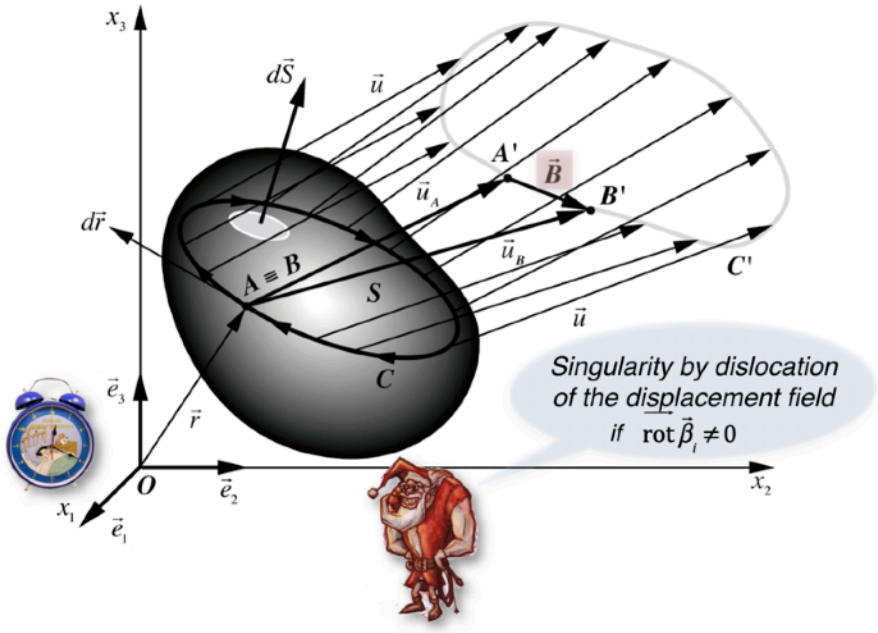


Figure 1.14 - *Singularity by dislocation of the displacement field*

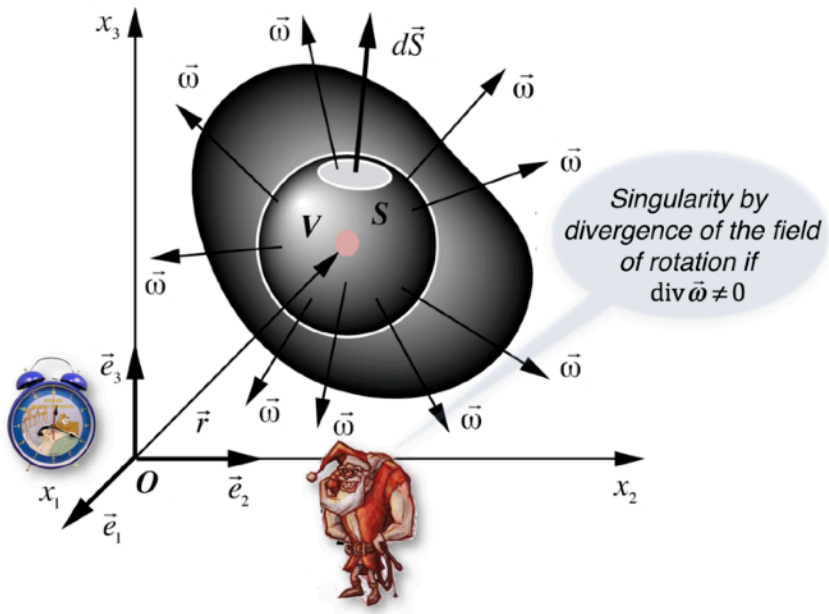


Figure 1.15 - *Singularity by divergence of the rotation field*

that there was no localization of the distortions, such as tears followed by a sliding of the jaws relative to each other, as well as *the topological compactness* of the medium, that is to say from a physical point of view the fact that it does not appear cavity formations or overlaps within the medium. In summary, the condition $\overrightarrow{\text{rot}}\,\vec{\beta}_i = 0$ ensures *the solidity of the medium.*

To find the meaning of the compatibility condition $\text{div}\,\vec{\omega} = 0$ for the rotation vector $\vec{\omega}$, we draw the rotation vector $\vec{\omega}$ on a closed surface S surrounding a volume V of solid (figure 1.15). The compatibility condition for the rotation vector $\vec{\omega}$ then stipulates that the flow of the

rotation field crossing the closed surface S is zero, which implies that *there is no divergent singularity of the rotation field* within the solid, such as that shown in the drawing in figure 1.15.

Contortions of a solid lattice

In a solid lattice, the fields $\vec{\beta}_i, \vec{\varepsilon}_i, \vec{\alpha}_i, \vec{\omega}, \tau$ represent the set of distortions, deformations, shears, rotations and volume expansions that the lattice unit cells undergo locally. If each unit cell is subjected to a field of distortion, which can vary from one cell to another, it must also appear effects of flexion and torsions on a more macroscopic scale of the solid medium, related to the lattice continuity.

These "curvatures" of the solid will be called the contortions of the lattice. In a geometro-compatible medium, they depend in fact only on deformations of the lattice, and are therefore deduced as spatial variations of the deformation field $\vec{\varepsilon}_i$ in the manner illustrated by the diagram in figure 1.16. A *contortion tensor* $\vec{\chi}_i$ then appears, which can be broken down by

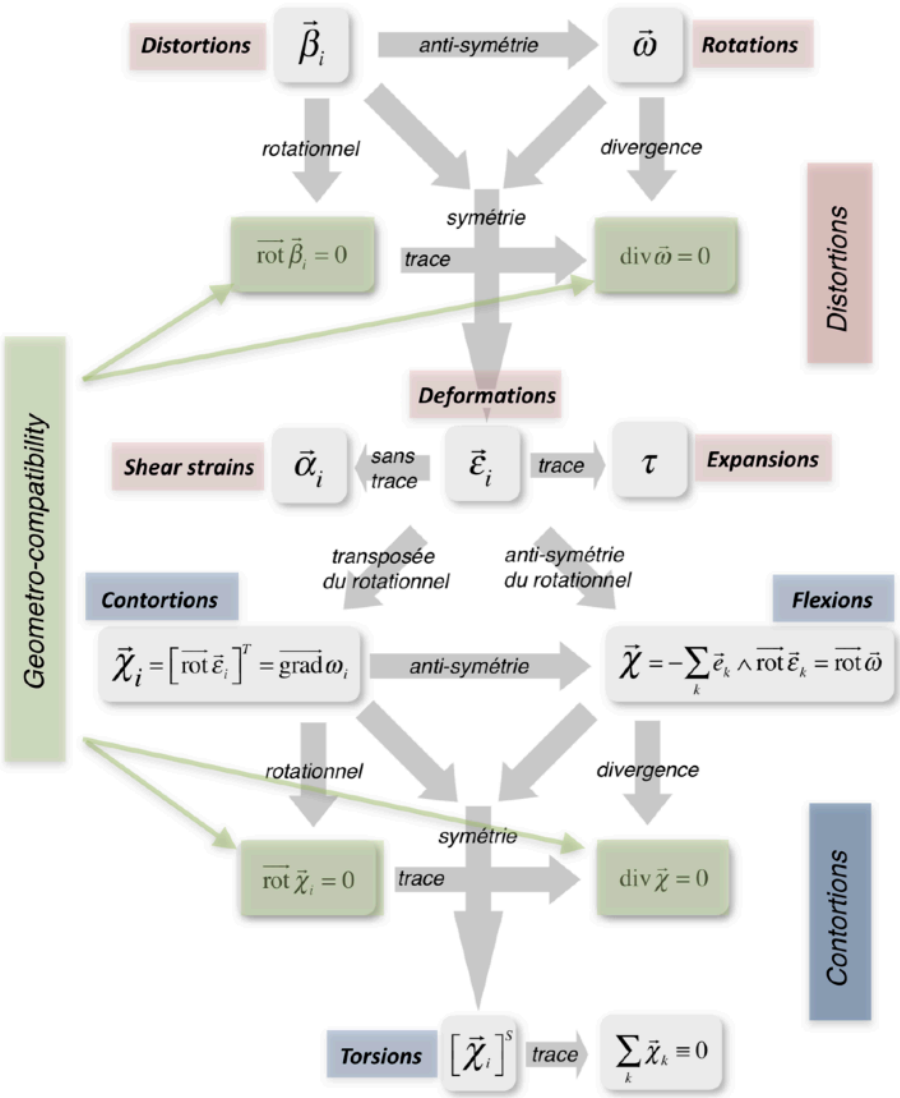


Figure 1.16 - *The complete system of distortions and contortions of a solid lattice*

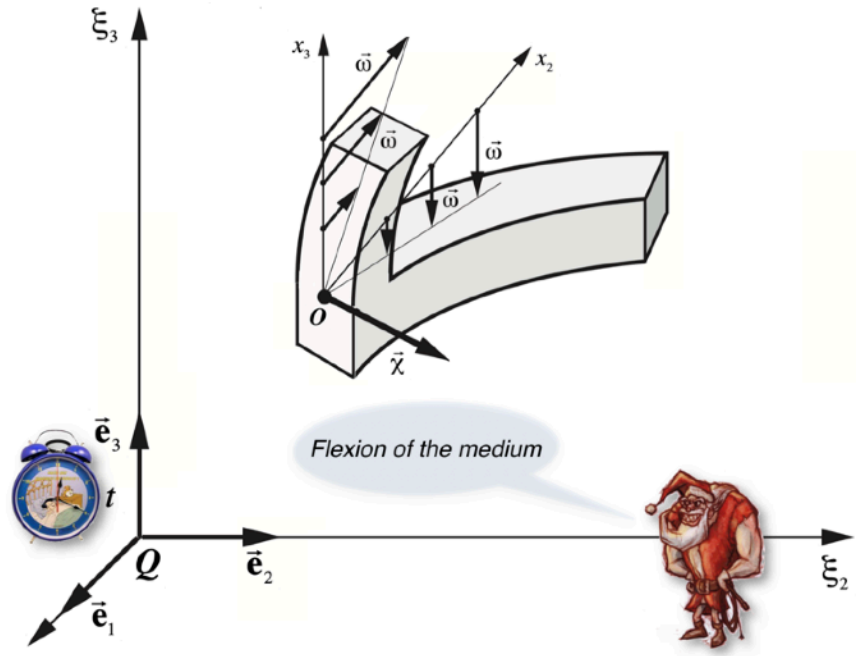


Figure 1.17 - *Flexion of the medium*

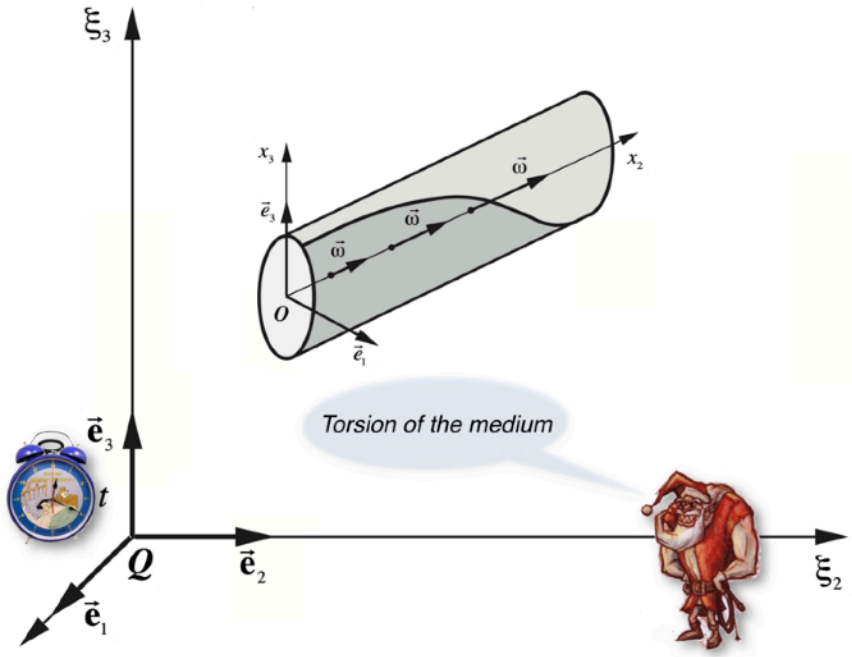


Figure 1.18 - *Torsion of the medium*

symmetries into a *flexion vector* $\vec{\chi}$ and a *transverse symmetric tensor (without trace) of torsion* $\left[\vec{\chi}_i\right]^S$.

In a geometro-compatible medium, the contortions of the lattice are also closely linked to the spatial derivatives of the rotation vector $\vec{\omega}$ as shown by the mathematical relationships giving the contortion tensor $\vec{\chi}_i$ and the flexion vector $\vec{\chi}$ in figure 1.16. These must therefore also

measure variations in rotations within the solid, such as *twists* or *bends*. We can therefore define more precisely the meaning of these tensors $\vec{\chi}_i$ and $\vec{\chi}$ using two typical examples of spatial variations of $\vec{\omega}$.

In the first example, a bent medium presents a rotation vector parallel to the axis Ox_3, and which increases in the direction of the axis Ox_2, as represented in figure 1.17. In this case, there is a non-diagonal component $\chi_{23} = \partial\omega_3 / \partial x_2 \neq 0$ of the tensor $\vec{\chi}_i$ which is not zero, and this component is associated with the *bending of the solid* as well illustrated by the figure 1.17.

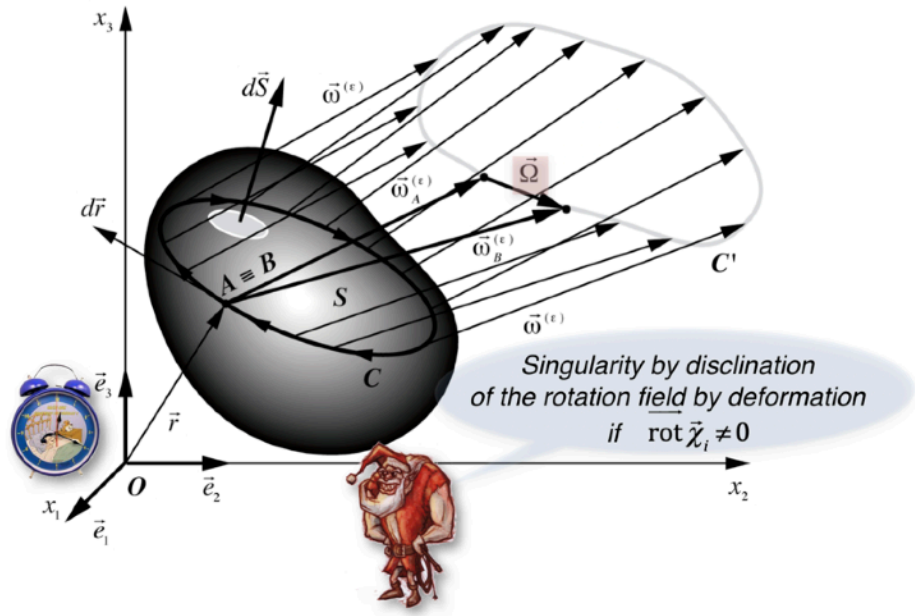


Figure 1.19 - *Singularity by disclination of the rotation field by deformation*

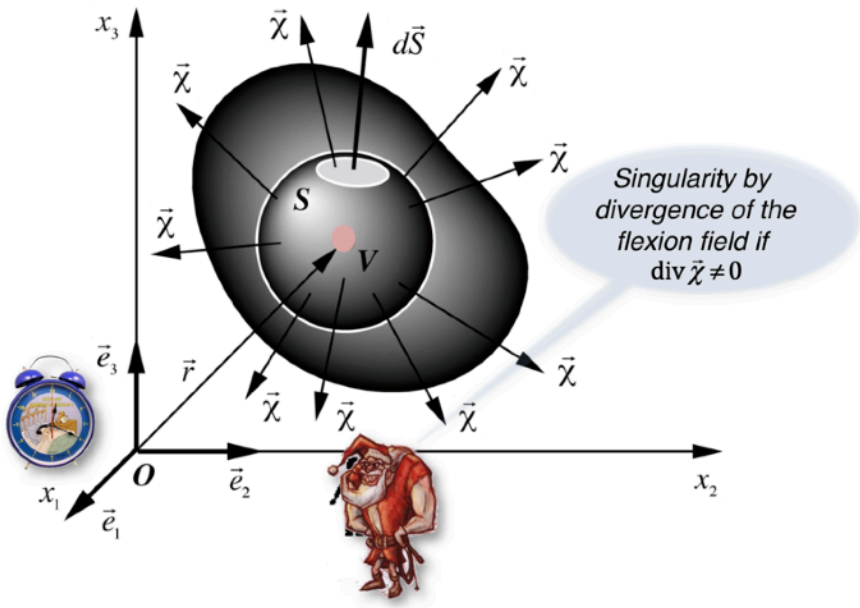


Figure 1.20 - *Singularity by divergence of the flexion field*

But in the case illustrated in figure 1.17, the vector $\vec{\omega}$ seems to turn around the axis Ox_1 so that there is *a non-zero flexion vector* $\vec{\chi} = -\overrightarrow{\mathrm{rot}}\,\vec{\omega}$ parallel to this axis as illustrated in figure 1.17.

In the second example (figure 1.18), we represent an increase along axis Ox_2 of a rotation vector parallel to axis Ox_2. For once it is a *diagonal component* $\chi_{22} = \partial\omega_2 / \partial x_2 \neq 0$ of the tensor $\vec{\chi}_i$ which necessarily becomes non-zero. And we see that this corresponds to a torsion of the solid medium.

In these two examples, it has been illustrated that spatial variations in the rotation vector $\vec{\omega}$ do indeed lead to bending or twisting of the solid medium. But in a geometro-compatible medium, these same bends and twists are also deductible directly from the strain tensor $\vec{\varepsilon}_i$.

Geometro-compatibility of lattice contortions

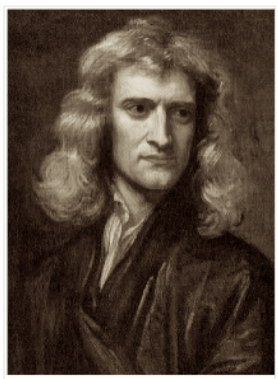
In the table of figure 1.16, we note that there are also two geometro-compatibility conditions for the contortion tensor $\vec{\chi}_i$ and the flexion vector $\vec{\chi}$ respectively, in a very similar way as in the case of the distortion tensor and the rotation vector.

The condition of compatibility $\overrightarrow{\mathrm{rot}}\,\vec{\chi}_i = 0$ of the contortion tensor $\vec{\chi}_i$ can be interpreted by considering a closed contour C within the medium and by plotting the local rotation vector $\vec{\omega}^{(\varepsilon)}$ deduced from the strain tensor along this contour (figure 1.19). The condition of compatibility implies then that the closure vector $\vec{\Omega}$, called *the Frank vector*, is zero, which means from a topological and physical point of view that *there are no discontinuities of rotations by deformation*, which are called *disclinations* in the medium.

The condition of compatibility $\mathrm{div}\,\vec{\chi} = 0$ of the flexion vector $\vec{\chi}$ is interpreted by considering a closed surface S surrounding a volume V of medium (figure 1.20). The compatibility condition for the flexion vector $\vec{\chi}$ then stipulates that the flow of the bending field crossing the closed surface S is zero, which implies that *there is no diverging singularity of the flexion field* within the solid, such as that shown in the drawing in figure 1.20.

Newtonian dynamics and Eulerian thermokinetics

It has been shown previously that a solid collection of particles in space can present a collective movement which corresponds to the global movements of translation, of global and local rotation and of deformation of the medium in the space of the observer, and that, in Euler coordinates, these movements are described by an average local speed $\vec{\phi}(\vec{r},t)$. To go further in the description and prediction of these movements, we must now introduce the physical principles to which the environment obeys.



Isaac Newton (1643-1727)

We suppose then that the solid lattice considered behaves in a Newtonian way in the absolute reference frame of ***GO***, in other words that the dynamics of the particles of the medium satisfy Newton's law $\vec{f} = m\vec{a}$ which implies that the acceleration $\vec{a}$ of a particle is related to the force $\vec{f}$ that we apply to it via t*he mass of inertia* m of the particle. In addition, we also admit that the physical

behaviors of the particle lattice obey the first two principles of thermodynamics, which have never been faulted, namely that energy is a conserved quantity, and that there exists a quantity called *entropy* which measures the state of disorder of the lattice.

We are going to introduce here rigorously the Newtonian dynamics and the thermokinetics of the lattice in Eulerian coordinates, starting from these three completely classic and well-known basic physical axioms. With this axiomatic and rigorous approach, we are led to define average quantities per site of the lattice, as well as sources and flows of physical quantities, which must satisfy three principles of continuity shown in figure 1.21.

Admitting the axiom that the individual movement of each particle of the lattice satisfies *a Newtonian dynamic*, namely the Newton's law $\vec{f} = m\vec{a}$, to the movement of each particle at velocity $\vec{\phi}$ must correspond a *momentum* and a *kinetic energy*. Per particle, according to Newtonian mechanics, the momentum is written $\vec{p} = m\vec{\phi}$ and the kinetic energy is written $e_{cin} = m\vec{\phi}^2/2$. These expressions of the momentum and the kinetic energy of a particle of the medium calls upon a conservative scalar physical quantity, specific to the particle: its *mass of inertia* or *inert mass* m. In Euler coordinates, we show that the conservation of the mass of inertia leads to a *principle of continuity for the mass of inertia* which is reported in figure 1.21.

This principle links the local temporal variation $\partial\rho/\partial t$ of the quantity of mass contained in the volume unit of the medium, which is called *the mass density* ρ of the medium, to: *(i)* the existence of a volume source of mass S_m, associated with creation or annihilation of mass within the medium and which is generally zero due to the principle of conservation of mass, *(ii)* the temporal variation $d\tau/dt$ of the scalar of volume expansion of the medium along the trajectory of its particles, *(iii)* the existence of *a source* S_n *of lattice sites* and *(iv)* the existence of a *mass transport flow* $\vec{J}_m$, i.e. a mass displacement by another physical process within the lattice, such as the self-diffusion which we will talk about later.



Sadi Carnot (1837-1894)

The movements of random fluctuations $\Delta\vec{v}_i$ of the medium particles corresponding to thermal agitation and the attractive or repulsive interactions that may exist between the particles of the medium must correspond respectively to *an internal thermal energy* and *an internal potential energy*. It is precisely the subject of the axiom of *the first principle of phenomenological thermodynamics,* which postulates the existence, for a given physical system, of a function U depending on the state of the system, called *the internal energy of the system*, which is such that, for any infinitesimal transformation of the system (any infinitesimal variation of one of the physical quantities of the system), we have the relation $dU = \delta W + \delta Q$, where dU represents the variation of the internal energy of the system, δQ represents the set of heat exchanges between the system and the outside world, and δW all of the work exchanges between the system and the outside world.

In the case of a solid lattice moving at velocity $\vec{\phi}$, the total energy is linked both to its internal energy and to its kinetic energy of global movement at velocity $\vec{\phi}$. In Euler coordinates, we show that the conservation of the energy of the deforming solid leads to *a principle of continuity*

for the total energy which is shown in figure 1.21. By expressing *the average internal energy* u and the average kinetic energy e_{cin} per lattice site, the continuity principle of energy states that the sum of the variations of internal energy u and kinetic energy e_{cin} taken along the trajectory of the particles depends on: *(i)* the existence of *an external work source* $S_{\vec{w}}^{ext}$ corresponding to a supply of mechanical energy from outside the medium, *(ii) the work flow* $\vec{J}_w$, i.e. exchanges of mechanical energy within the medium, *(iii) the heat flow* $\vec{J}_q$, that is to say heat exchanges within the medium, and *(iv) the source* S_n *of lattice sites*, that is to say the creation or annihilation of a certain number of lattice sites per unit of time.

The *second principle of phenomenological thermodynamics* postulates the existence, for a given physical system, of a function S depending on the state of the system, called *the entropy of the system*. This function in fact characterizes the state of disorder reigning within the system, and it is such that any infinitesimal transformation of the system satisfies the relation $dS \geq \delta Q / T$ where dS represents the variation of the entropy of the system, δQ represents the set of heat exchanges between the system and the outside world and T is the temperature of the system.

In Euler coordinates, we show that the second principle of phenomenological thermodynamics leads to *a principle of continuity for the entropy* which is reported in figure 1.21.

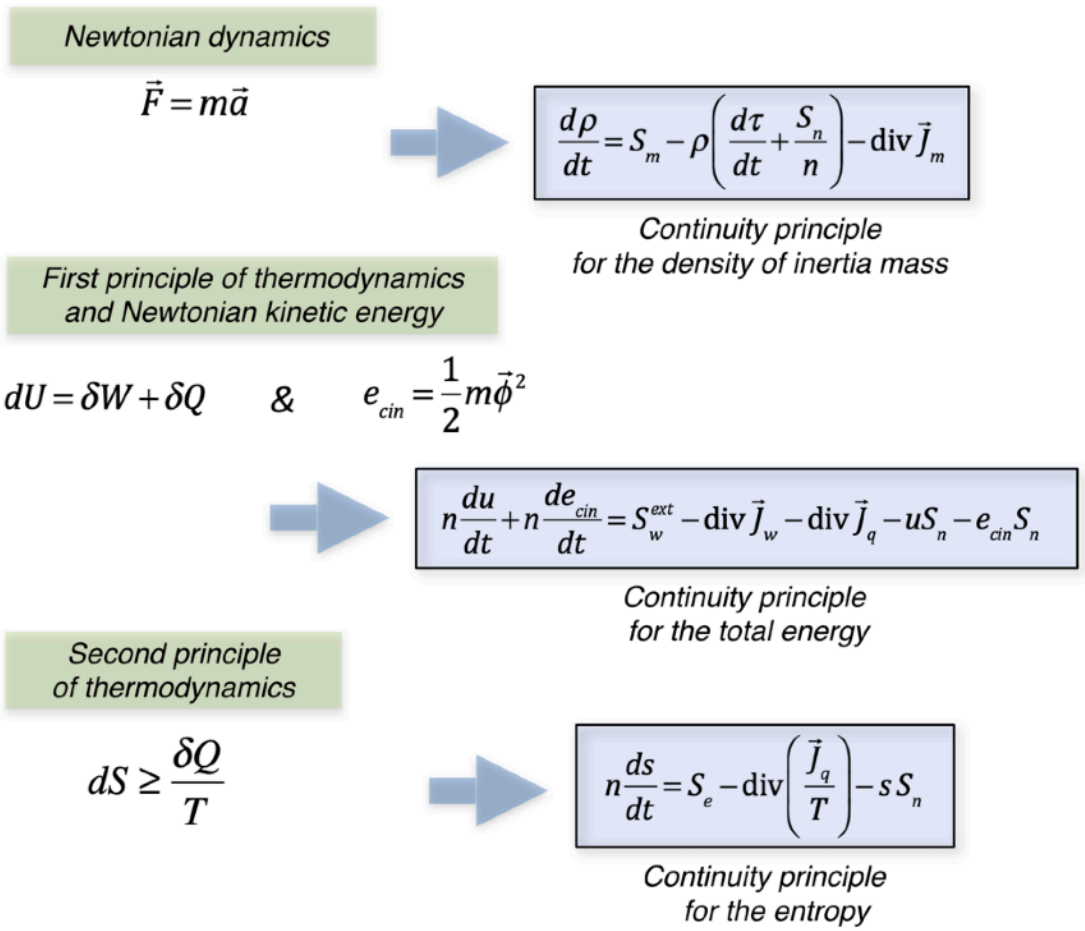


Figure 1.21 *- The three Eulerian continuity principles deduced from the Newtonian dynamics and from the two basic principles of the thermodynamics*

By expressing the average entropy s per lattice site, the principle of continuity of entropy stipulates that the variation of entropy s taken along the trajectory of particles depends on: *(i)* the existence of a volume source of entropy S_e , that is to say, the local creation of entropy within the medium, *(ii) the heat flow* $\vec{J}_q$ within the medium, *(iii)* the temperature T of the medium which characterizes the state of thermal agitation of the medium particles and *(iv) the source* S_n *of lattice sites*.

These three physical principles are absolutely essential in solid media, and they are *the only fundamental physical principles absolutely necessary* for a complete description of Newtonian geometro-dynamics and of phenomenological thermodynamics of deformable media in Euler coordinates.

Physical properties specific to the medium

The equations of geometro-kinetics (figure 1.12) and geometro-compatibility (figure 1.16), as well as the three principles of continuity deduced from Newton's laws and phenomenological thermodynamics (figure 1.21) are the basic concepts for the treatment of deformable solid media in Euler coordinates, and remain the same whatever the solid considered. But we know very well that the physical properties and the macroscopic behaviors that we can observe on a solid medium can be very different from one medium to another. These physical properties are called *the phenomenological properties of the medium*. We will therefore now discuss the most important phenomenological physical properties of a solid lattice, namely *the mechanical properties of the lattice*, such as *elasticity* and *anelasticity*, and *the mass transport properties* within the lattice, such as *self-diffusion*.

It is common experience that you can bend a hacksaw blade strongly without great effort, and that it resumes all its straightness when you release the force that you applied to it. This typical phenomenological behavior is called *the elasticity of the solid*. The elasticity of the medium is due to the internal bonding forces between particles of the solid, so that bending the saw blade amounts to storing internal energy in the medium, in fact in the bonds between its particles, energy which is then used to return the blade to its original shape when released. Thus, to introduce this property of elasticity in the Eulerian equations of the medium, one will use *the average internal energy* u *per site of lattice* which one defined in the preceding paragraph, and one will express that this one depends on the deformations applied to the solid, in other words elastic deformation fields $\vec{\beta}_i^{\,el}$, $\vec{\varepsilon}_i^{\,el}$, $\vec{\alpha}_i^{\,el}$, $\vec{\omega}^{el}$, τ^{el} appearing in the solid when it is deformed. We express this situation by saying that internal energy u is a function $u(\vec{\beta}_i^{\,el},s)=u(\vec{\varepsilon}_i^{\,el},\vec{\omega}^{el},s)=u(\vec{\alpha}_i^{\,el},\vec{\omega}^{el},\tau^{el},s)$ depending on the state of elastic strain tensors, but also on the local entropy of the network, via *the average entropy* s *per lattice site.* This dependence on entropy reflects the effect on internal energy of the existence of a spatial and kinetic disorder within the medium, and in particular the effect of thermal agitation of the particles under the effect of heat.

When a solid is deformed, elasticity is an immediate response of the solid. But there can sometimes be another response from the solid which is generally added to the elastic response and which is delayed in time compared to the solicitation of the solid, but which is also recoverable with delay when the solicitation is released. Such a response of the solid is called *the anelasticity of the solid*. While the elastic response does not release thermal energy into the

solid, the anelastic response releases thermal energy when activated, and is said to be a dissipative process.

It is this type of process which dissipates the vibration energy of certain metals, such as cast iron for example, which are used in industry precisely for this property of damping vibrations and noise. When anelastic deformation phenomena are activated in a solid, these also store internal energy in the solid, energy which is then used by the solid to restore its initial shape, as in the case of elasticity. Anelastic deformations $\vec{\beta}_i^{an}, \vec{\varepsilon}_i^{an}, \vec{\alpha}_i^{an}, \vec{\omega}^{an}$ can therefore appear in a solid, which, as in the case of elastic deformations, also modify the function of the internal energy of the solid $u(\vec{\beta}_i^{el}, \vec{\beta}_i^{an}, s) = u(\vec{\varepsilon}_i^{el}, \vec{\varepsilon}_i^{an}, \vec{\omega}^{el}, \vec{\omega}^{an}, s) = u(\vec{\alpha}_i^{el}, \vec{\alpha}_i^{an}, \vec{\omega}^{el}, \vec{\omega}^{an}, \tau^{el}, s)$. This means that the internal energy of the solid is a function of the state of both elasticity and anelasticity of the solid. It will be noted here that, for the sake of simplification, it has been assumed that it does exist anelastic volume expansion, that is to say that $\tau^{an} = 0$.

In principle, the existence of a non-zero *source* S_n *of lattice sites* would violate the Newtonian principle of mass conservation, unless there is a phenomenon of *self-diffusion by intrinsic point defects within the lattice*. An *intrinsic point defect of the lacunar type*, simply called *a vacancy*, is a site of the particle solid lattice which has no particles (figure 1.22). It is therefore a *"hole"* in the lattice. An *intrinsic point defect of interstitial type*, called simply *a self-interstitial*, is a particle which is in the solid lattice, but which does not occupy a regular substitutional site of this lattice (figure 1.22). It is therefore an *"additional"* particle in the lattice.

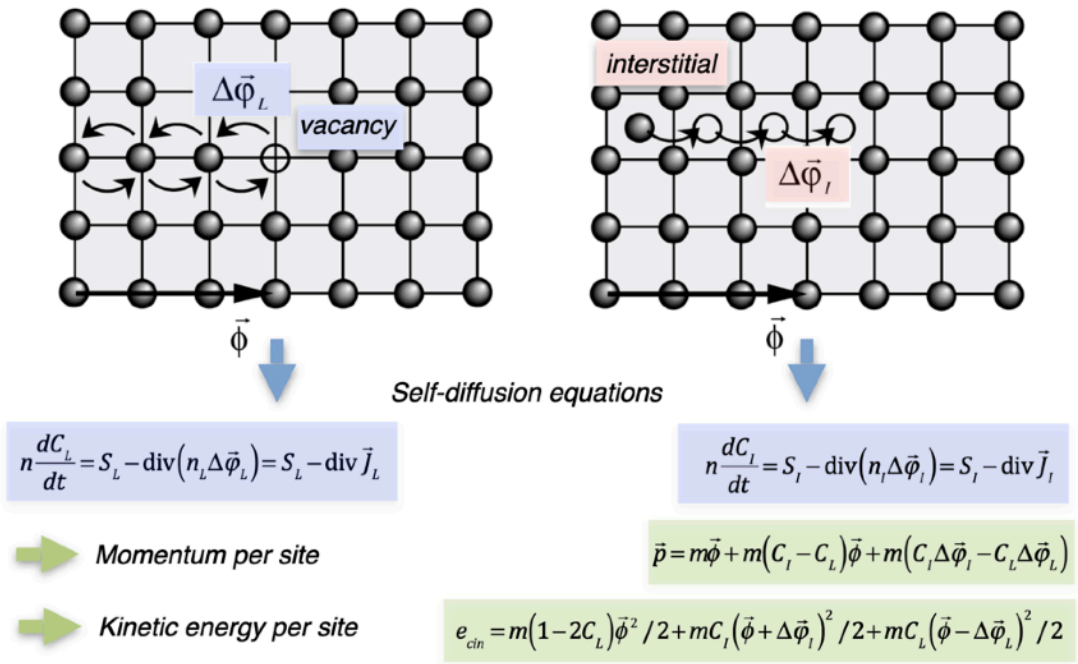


Figure 1.22 *- Self-diffusion by vacancy and by self-interstitial*

It is simple to understand how the presence of such intrinsic point defects can lead to the existence of mass transport phenomena by *lacunar and / or interstitial self-diffusion*. These two mass transport mechanisms can be illustrated in a lattice moving at absolute velocity $\vec{\phi}$ in the

space of the external observer ***GO*** (figure 1.22). The movement of a vacancy at velocity $\Delta\vec{\varphi}_L$ relative to the lattice in a given direction leads to a relative mass flow in the opposite direction, at velocity $-\Delta\vec{\varphi}_L$, while the movement of a self-interstitial at velocity $\Delta\vec{\varphi}_I$ relative to the lattice in a given direction causes a mass flow in the same direction, at velocity $\Delta\vec{\varphi}_I$.

To transcribe mathematically the existence of these point defects, we must first introduce the volume densities of vacancies and self-interstitials, i.e. the number $n_L = n_L(\vec{r},t)$ of vacancies and $n_I = n_I(\vec{r},t)$ of self-interstitials per volume unit of lattice. From these densities, it is possible to define *atomic concentrations of vacancies and auto-interstitials* with respect to the density n of network sites by relations $C_L = n_L / n$ and $C_I = n_I / n$. Contrary to appearances, there is a certain asymmetry between vacancies and auto-interstitials, which is expressed in the fact that the maximum atomic concentration of vacancies is always limited to 1, when all the sites of the lattice are locally unoccupied, while the atomic concentration of auto-interstitials depends on the number of interstitial sites accessible in each unit cell of a lattice with a given structure, and on the number of interstitials that it is possible to fit on each of these sites.

We can also introduce the *diffusion flows* $\vec{J}_L$ *and* $\vec{J}_I$ *of vacancies and interstitials* with respect to the lattice, defined by relations $\vec{J}_L = nC_L\Delta\vec{\varphi}_L$ and $\vec{J}_I = nC_I\Delta\vec{\varphi}_I$, and which measure the number of vacancies and interstitials that cross the unit surface per unit of time within the lattice. We can then deduce *the self-diffusion equations* reported in figure 1.22, which allow to calculate the temporal variations of the atomic concentrations of vacancies and auto-interstitials along the trajectory of the medium, on the one hand by the introduction of *volume sources of vacancies* S_L *and interstitials* S_I associated with the number of creations and annihilations of vacancies and interstitials per unit of time and per unit of volume, and on the other hand the introduction of *fluxes* $\vec{J}_L$ *and* $\vec{J}_I$ *of vacancies and interstitials* associated with the number of mobile vacancies and mobile interstitials crossing per unit time a unit surface within the lattice. The principle of mass conservation is then satisfied if the source of lattice sites is directly linked to the local creations and annihilations of vacancies and interstitials, that is to say linked to the sources S_L and S_I of vacancies and interstitials, via the relationship $S_n = S_L - S_I$.

In an elastic and anelastic network with self-diffusion, it is intuitively clear that the atomic concentrations C_L and C_I of vacancies and auto-interstitial must also influence the energy state of the lattice, so that the internal energy state function has to be written now as a function $u(\vec{\beta}_i^{\,el}, \vec{\beta}_i^{\,an}, C_L, C_I, s) = u(\vec{\varepsilon}_i^{\,el}, \vec{\varepsilon}_i^{\,an}, \vec{\omega}^{el}, \vec{\omega}^{an}, C_L, C_I, s) = u(\vec{\alpha}_i^{\,el}, \vec{\alpha}_i^{\,an}, \vec{\omega}^{el}, \vec{\omega}^{an}, \tau^{el}, C_L, C_I, s)$ of the set of thermodynamic variables that characterize the state of the medium.

The presence of vacancies and self-interstitials in the lattice will modify the expressions of kinetic energy and momentum expressed as an average value per particle or per lattice site, as shown in figure 1.22.

The function $u(\vec{\alpha}_i^{\,el}, \vec{\alpha}_i^{\,an}, \vec{\omega}^{el}, \vec{\omega}^{an}, \tau^{el}, C_L, C_I, s)$ which characterizes the internal energy state of the solid lattice is a phenomenological quantity of the considered medium, in the sense that it must be established for each medium and that it is specific to each medium. It is essentially this which will control the space-time behavior of the medium via *thermodynamic potentials*. Indeed, if we express the temporal variation of the internal energy along the trajectory of the medium, we obtain an equation which is called *the thermokinetic equation* of the medium, which is expressed from the relation $u(\vec{\alpha}_i^{\,el}, \vec{\alpha}_i^{\,an}, \vec{\omega}^{el}, \vec{\omega}^{an}, \tau^{el}, C_L, C_I, s)$ as shown in figure 1.23.

This thermokinetic equation reveals *mechanical potentials* $\vec{s}_k, \vec{m}, p, \vec{s}_k^{\,cons}, \vec{m}^{cons}$ called *the stress tensors* $\vec{s}_k$ *and* $\vec{s}_k^{\,cons}$ conjugated respectively to the elastic and anelastic shear tensors $\vec{\alpha}_i^{\,el}$ and $\vec{\alpha}_i^{\,an}$, *the moment vectors* $\vec{m}$ *and* $\vec{m}^{cons}$ conjugated respectively to the vectors of elastic and anelastic rotation $\vec{\omega}^{el}$ and $\vec{\omega}^{an}$, *the pressure* p conjugated to the scalar of volume expansion τ^{el}. These mechanical potentials represent the internal mechanical forces inside the medium, which tend to eliminate the deformations of the medium to restore the undeformed state of the solid medium. These mechanical potentials can be deduced from one another according to the graph shown in figure 1.23.

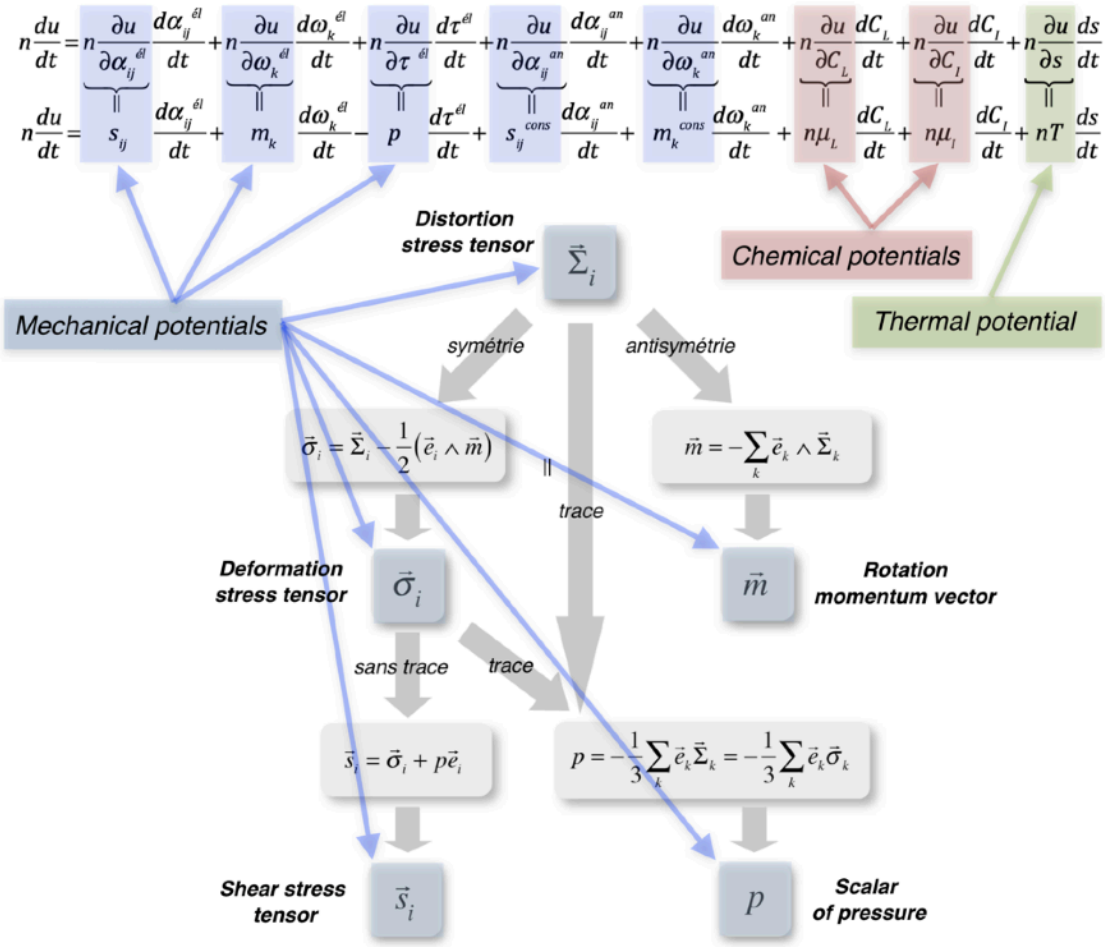


Figure 1.23 *- Thermokinetics equations*

The thermokinetic equation also reveals *the chemical potentials* μ_L *and* μ_I, conjugated with the atomic concentrations C_L and C_I of vacancies and auto-interstitials respectively. These chemical potentials represent in fact the internal chemical forces acting on the vacancies and the auto-interstitials within the medium, which tend to eliminate spatial variations in atomic concentrations C_L and C_I of vacancies and auto-interstitials, in order to restore the chemical equilibrium state inside the solid medium.

Finally, the thermokinetic equation still reveals a significant quantity of the medium, namely its temperature T, which is conjugated with its entropy s, and which measures the thermal agitation within the medium.

Potentialities of the Eulerian representation of deformable media

To end this chapter, we represent by a graph in figure 1.24 all the potentialities of the Eulerian representation to describe the spatio-temporal evolution of deformable Newtonian media, potentialities which are developed in detail in my first book written in French.

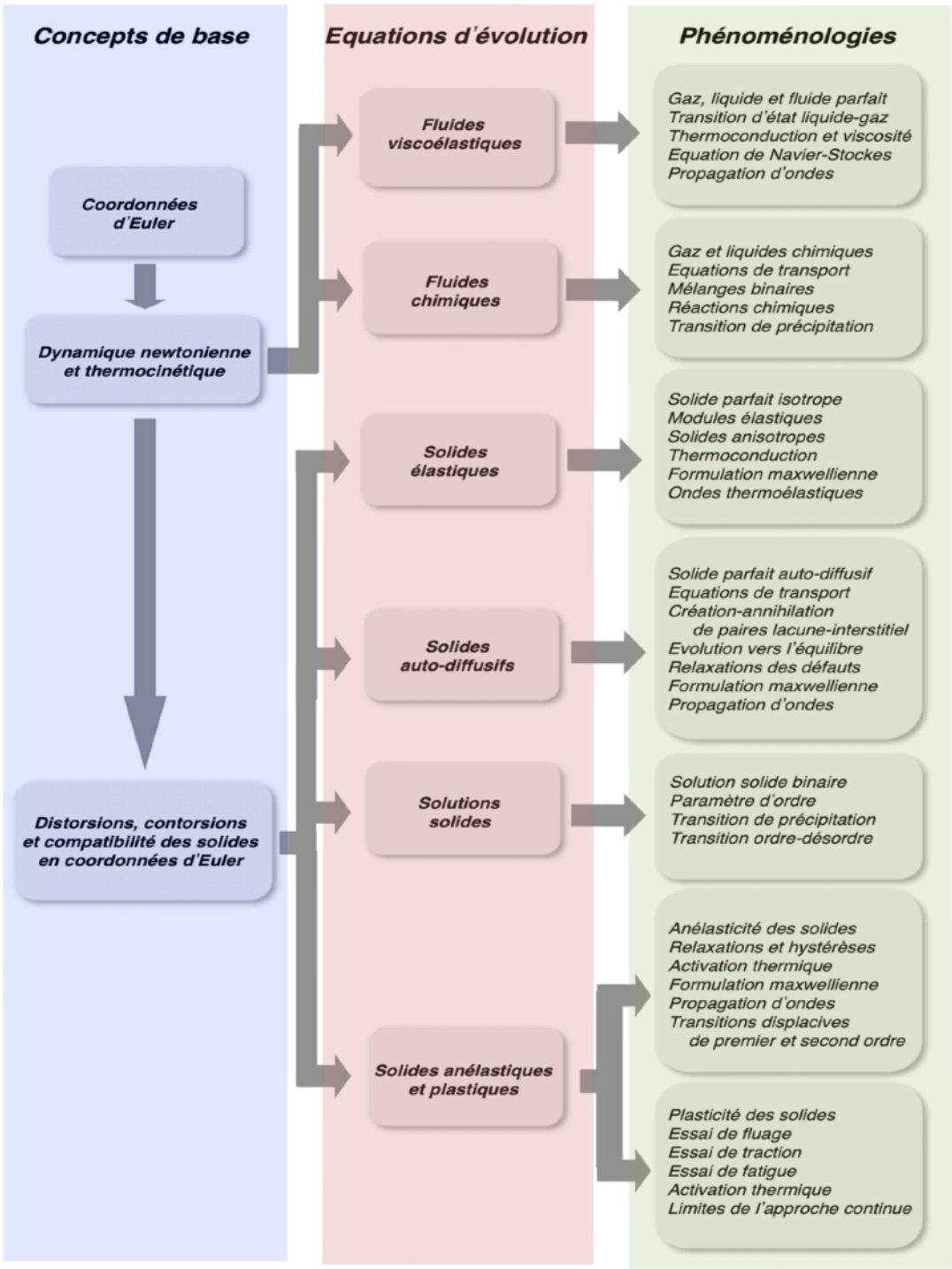


Figure 1.24 - *The potentialities of the Eulerian representation of Newtonian deformable media*